A CANONICAL FORM FOR
NONLINEAR SYSTEMS

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The concepts of transformation and canonical form have been used in analyzing linear systems. In this article we extend these ideas to nonlinear systems. A coordinate system and a corresponding canonical form are developed for general nonlinear control systems. Their usefulness is demonstrated by showing that every feedback linearizable system becomes a system with only feedback paths in the canonical form. For control design involving a nonlinear system, one approach is to put the system in its canonical form and approximate by that part having only feedback paths.

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## I. Introduction

It is well known that every linear system can be transformed by state variable changes and state feedback to a canonical form consisting of multiple series of integrators. The number of integrators involved in each series is called a Kronecker index, and a linear system is completely characterized by its Kronecker indices up to the feedback transformation. We extend this result to a large class of nonlinear systems. In a setting of great generality, the concept of nonlinear feedback equivalence can be defined as follows. A system

$$\dot{x} = f(x,u), \; x \in R^n, \; u \in R^m \tag{1}$$

is said to be feedback equivalent to another system

$$\dot{y} = g(y,v), \; y \in R^n, \; v \in R^m \tag{2}$$

if there exists a pair of mappings

$$T : R^n \rightarrow R^n \tag{3}$$

$$W : R^n \times R^m \rightarrow R^m \tag{4}$$

such that (i) T is invertible on a neighborhood near the origin of $R^n$ and W is invertible with respect to the second variable in $R^m$, and (ii) for every solution $(x(t), u(t))$ of equation (1), the induced pair of time functions $(T(x(t)), W(x(t), u(t)))$ is a solution of equation (2) by substituting $T(t)$ for $y(t)$ and $W(t)$ for $v(t)$. The mappings T and W are of course the nonlinear generalization of the state variable changes and the state feedback, respectively, in linear theory.

According to a result obtained by the authors and G. Meyer [1] (also independently by Jakubczyh and Respondeck [2]), a system of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\, u_i \tag{5}$$

is feedback equivalent to a linear system

$$\dot{y} = Ay + \sum_{i=1}^{m} b_i v_i \tag{6}$$

if and only if the following conditions are satisfied (with possible re-numbering of $g_i$):

(a) The set $C = \{g_1, [f,g_1], \ldots, (ad^{\kappa_1 - 1} f, g_1), g_2, [f,g_2], \ldots, (ad^{\kappa_2 - 1} f, g_2), \ldots, g_m, [f,g_m], \ldots, (ad^{\kappa_m - 1} f, g_m)\}$ spans $R^n$,

(b) The sets $C_j = \{g_1, [f,g_1], \ldots, (ad^{\kappa_j - 2} f, g_1), g_2, [f,g_2], \ldots, (ad^{\kappa_j - 2} f, g_2), \ldots, g_m, [f,g_m], \ldots, (ad^{\kappa_j - 2} f, g_m)\}$ are involutive for $j = 1, 2, \ldots, m$,

(c) The span of each $C_j$ is identical with the span of $C_j \cap C$, and

(d) $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_m$,

where $\kappa_1, \kappa_2, \ldots, \kappa_m$ are Kronecker indices for the linear system (6).

Thus, the importance of Kronecker indices has been extended by this result to nonlinear systems. Here, however, they play a dual role. In addition to controllability span, they also characterize the involutivity of a nonlinear system, which has no counterpart in linear systems.

This result has not only generated a fair amount of interest in the analytical problem of feedback linearization (see [1]), but also has been applied to the design of nonlinear controllers [3]. A recent successful flight test with the UH-1H helicopter by George Mayer at NASA Ames illustrates the practicality of our results. In experimenting with such designs, we are increasingly concerned with that part of a system's dynamics which is not linearizable through feedback transformation. We feel that a measure of the "gap" between a nonlinear system and a linear equivalent is needed for robustness analysis of such designs.

In this article we report our recent progress in this direction. We shall present a special coordinate system for the state space of a nonlinear system, to be called the s-coordinate system. These coordinates are derived on the basis of the intrinsic geometry of a system. In terms of them, a canonical form of a nonlinear system is then established. The usual Taylor series expansion is replaced by an expansion in terms of the important Lie brackets. This form is especially useful for studying feedback linearizability. We illustrate this fact by showing that every feedback linearizable system becomes a system with only feedback paths in s-coordinates. We emphasize that the s-coordinates can be produced for a general class of nonlinear control systems of which the feedback linearizable ones form a small (but important) subclass.

## II. s-Coordinate System.

Let $X_1, X_2, \ldots, X_n$ be an arbitrary sequence of linearly independent vector fields in a neighborhood $U \subset R^n$ of the origin. In this section we describe a local coordinate system for U which is based on the intrinsic geometry of $X_i$. We also assume throughout that the vector fields are analytic.

We first define our notation and make a simple geometric observation. For each vector field $X_k$, let $\phi_k : V_k \to R^n$ denote the one-parameter group action generated by $X_k$, where $V_k$ is an open set in $R \times U$ and includes $\{0\} \times U$; that is,

$$\frac{\partial}{\partial t} \phi_k(t,p) = X_k(\phi_k(t,p))$$

and

$$\phi_k(o,p) = p$$

for $(t,p)\varepsilon V_k$. Given any point $p\varepsilon U$, the mapping $\phi_k(\cdot,p)$ defines an integral curve of $X_k$. More generally, if $S_m$ is an m-dimensional manifold in U with the property that $X_k(p)$ does not belong to the tangent space of $S_m$ at p for all $p\varepsilon S_m$, then the union of all the integral curves $\phi_k(\cdot,p)$, $p\varepsilon S_m$, defines an (m+1)-dimensional manifold $S_{m+1}$, and, obviously, $X_k(q)$ is a tangent vector of $S_{m+1}$ at $q\varepsilon S_{m+1}$.

Next we construct a sequence of manifolds associated with $X_k$. We define

$$S_o = \{0\varepsilon R^n\} \tag{7}$$

and $$S_k = \{\phi_k(t,p)|(t,p)\varepsilon V_k,\ p\varepsilon S_{k-1}\} \cap U$$

where $k = 1,2,\ldots,n$. From the assumption that $X_k$ are linearly independent on U and our previous observation, it follows that each $S_k$ is a k-dimensional manifold resulting from the union of the integral curves of $X_k$. Moreover, these manifolds form an increasing sequence as

$$S_o \subset S_1 \subset \ldots \subset S_n \subset U.$$

We now show that, with the aid of $S_k$, an n-tuple of numbers can be uniquely assigned to every point $p\varepsilon S_n$ which turns out to be a local coordinate system for $S_n$. For a point $p\varepsilon S_n$, there is a unique integral curve of $X_n$ passing through p. Suppose this curve intersects $S_{n-1}$ at a point $q\varepsilon S_{n-1}$. A unique parameter $s^n$ is thus determined by the equation $\phi_n(s^n,q)=p$. Considering now the point $q\varepsilon S_{n-1}$, there is an integral curve of $X_{n-1}$ passing through q and intersection $S_{n-2}$ at a point $r\varepsilon S_{n-2}$. Another unique parameter $s^{n-1}$ is determined by the equation $\phi_{n-1}(s^{n-1},r)=q$. Continuing this process, we finally reach the origin of $R^n$ with the integral curve of $X_1$ at the n-th step. The result is a set of n parameters $(s^1,s^2,\ldots,s^n)$ which is uniquely associated with the starting point p and satisfies the equation

$$\phi_n(s^n,(\phi_{n-1}(s^{n-1},(\ldots(\phi_1,(s^1,0))\ldots) = p. \tag{8}$$

It is not difficult to see that this mapping $F : S_n \to R^n$, $p_1 \to (s^1,\ldots,s^n)$ defines an analytic coordinate system for $S_n$. This is called the <u>s-coordinate system</u> generated by the vector fields $\{X_k\}_{k=1}^n$.

In the following, we observe some useful properties of the s-coordinates.

<u>Lemma 1</u>. In terms of the s-coordinates, the manifolds $S_k$ defined by (7) are simply linear subspaces of $R^n$:

$$S_k = \{s=(s^1,s^2,\ldots,s^n) \in R^n | s^m=0,\ k+1 \le m \le n\}.$$

And on each $S_k$, the vector field $X_k$ takes the simple form

$$X_k\Big|_{S_k} = \frac{\partial}{\partial s^k},$$

or in the form of a column vector

$$X_k\Big|_{S_k} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k^{th} \text{ place}, \tag{9}$$

where $X_k|_{S_k}$ denotes the restriction of $X_k$ on $S_k$.

This result follows immediately from the definitions of s-coordinates and $S_k$. For a detailed proof, the reader is referred to Spivak [4]. It must be noted that the simple expression (9) does not hold for the points not on $S_k$. A complete expression for $X_k$ is given in the following Lemma.

<u>Lemma 2</u>. In terms of s-coordinates and $S_k$, the vector field $X_k$ can be expressed as

$$X_k = e_k + \sum_{i=1}^{\infty} \sum_{j=k+1}^{n} \frac{(s^j)^i}{i!} (ad^i X_j, X_k)\Big|_{S_{j-1}} \tag{10}$$

where $k = 1, 2, \ldots, n$, and $e_k = (\delta_{1k}, \ldots, \delta_{nk})^T$ with $\delta_{ij}$ being the Kronecker delta.

Proof. Let $X_k = \begin{pmatrix} X_k^1 \\ X_k^2 \\ \vdots \\ X_k^n \end{pmatrix}$ .

By a Taylor series expansion of $X_k(s)$ with respect to $s^n$, we have

$$X_k(s) = \begin{pmatrix} X_k^1 \\ X_k^2 \\ \vdots \\ X_k^n \end{pmatrix}(s^1, \ldots, s^{n-1}, 0) + \sum_{i=1}^{\infty} \frac{s^i}{i!} \frac{\partial^i}{\partial (s^n)^i} \begin{pmatrix} X_k^1 \\ X_k^2 \\ \vdots \\ X_k^n \end{pmatrix}(s^1, \ldots, s^{n-1}, 0) \tag{11}$$

From Lemma 1 it is easy to see that the Lie bracket

$$\begin{aligned}
[X_n, X_k] &= \frac{\partial X_k}{\partial s} X_n - \frac{\partial X_n}{\partial s} X_k \\
&= \begin{pmatrix} \frac{\partial X_k^1}{\partial s^1} & \cdots & \frac{\partial X_k^1}{\partial s^n} \\ \vdots & & \\ \frac{\partial X_k^n}{\partial s^1} & \cdots & \frac{\partial X_k^n}{\partial s^n} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ \\ 0 \end{pmatrix}
\end{aligned}$$

$$= \frac{\partial}{\partial s^n}\begin{pmatrix} X_k^1 \\ \vdots \\ X_k^n \end{pmatrix}$$

We set $(ad^0 X_n, X_k) = X_k$, $(ad^1 X_n, X_k) = [X_n, X_k]$, $(ad^2 X_n, X_k) = (ad^1 X_n (ad^1 X_n, X_k))$, etc.

The same computation leads to

$$(ad^i X_n, X_k) = \frac{\partial^i X_k}{\partial (s^n)^i} .$$

Using the notation of restriction, we rewrite (11) as

$$X_k(s) = X_k\Big|_{S_{n-1}} + \sum_{i=1}^{\infty} \frac{(s^n)^i}{i!} (ad^i X_n, X_k)\Big|_{S_{n-1}}$$

Similarly, $X_k\Big|_{S_{n-1}}$ can be further expanded with respect to $s^{n-1}$ as

$$X_k\Big|_{S_{n-1}} = X_k\Big|_{S_{n-2}} + \sum_{i=1}^{\infty} \frac{(s^{n-1})^i}{i!} (ad^i X_{n-1}, X_k)\Big|_{S_{n-2}} ,$$

continuing this process until we have

$$X_k\Big|_{S_{k+1}} = X_k\Big|_{S_k} + \sum_{i=1}^{\infty} \frac{(s^{k+1})^i}{i!} (ad^i X_{k+1}, X_k)\Big|_{S_k} .$$

From Lemma 1, $X_k\Big|_{S_k} = e_k$. Therefore, combining all the expressions, we obtain

$$X_k(s) = e_k + \sum_{i=1}^{\infty} \sum_{j=k+1}^{n} \frac{(s^j)^i}{i!} (ad^i X_j, X_k)\Big|_{S_{j-1}} . \quad \square$$

We have so far only assumed that the $X_k$ are linearly independent on U. Additional assumption of involutiveness will further simplify the expression (10). Recall that a set of vector fields is <u>involutive</u> if the Lie bracket of any two vector fields is a linear combination of the elements in the set with coefficients being scalar functions.

<u>Lemma 3</u>. If the vector fields $X_k, X_{k+1}, \ldots, X_n$ are involutive, then $X_k$ assumes the form $X_k = (0, \ldots 0, X_k^k, \ldots, X_k^n)^T$ in the s-coordinates.

The proof of this statement is straightforward. We observe that for $j = k+1, \ldots, n$

$$(ad^i X_j, X_k) = \alpha^n(s)X_n + \alpha^{n-1}(s)X_{n-1} + \ldots + \alpha^k(s)X_k$$

from the assumption of involutiveness, where $\alpha(s)$ are scalar functions of s. Restricting to $S_{j-1}$, we have

$$(ad^i X_j, X_k)\Big|_{S_{j-1}} = \alpha^n X_n\Big|_{S_{j-1}} + \cdots + \alpha^{j-1} X_{j-1}\Big|_{S_{j-1}} + \alpha^k X_k\Big|_{S_{j-1}}$$

$$= \alpha^n e_n + \alpha^{n-1} e_{n-1} + \cdots + \alpha^{j-1} e_{j-1} + \alpha^{j-2} X_{j-2}\Big|_{S_{j-1}} + \cdots$$

$$+ \alpha^k X_k\Big|_{S_{j-1}}$$

We can further apply Lemma 2 and expand the vector fields $\alpha^{j-2}X_{j-2}, \ldots, \alpha^k X_k|_{S_{j-1}}$ to see that these terms must also have vanishing entries above the k-th place so as to have the form

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{pmatrix} \rightarrow \text{k-th place} \tag{12}$$

We conclude that each $(ad^i X_j, X_k)|_{S_{j-1}}$ has the form (12), and thus Lemma 3 is proved. □

If we assume a stronger condition that

$$[X_j, X_k] = 0 \tag{13}$$

for $j = k+1, \ldots, n$, then $X_k = e_k$. If (13) holds for all $1 \le j, k \le n$, then a classical result follows (see Spivak [4]): $X_k = e_k$, $k = 1, 2, \ldots, n$.

In later developments, we also need an expression for a general vector field Y on U in terms of the s-coordinate system generated by $X_1, X_2, \ldots, X_n$. The general expression is given in the next Lemma without proof.

<u>Lemma 4</u>. Let Y be an arbitrary vector field on U. In the s-coordinates generated by a sequence of vector fields $X_1, X_2, \ldots, X_n$, Y can be expressed as

$$Y(s) = Y(0) + \sum_{i=1}^{\infty} \sum_{j=1}^{n} \frac{(s^j)^i}{i!} (ad^i X_j, Y)\Big|_{S_{j-1}} \tag{14}$$

## IV. Canonical Form

We now consider the problem of establishing a canonical form for non-linear systems

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \ x \varepsilon R^n, \tag{15}$$

where the vector fields f and $g_i$ are analytic.

We first construct a sequence of n linearly independent vector fields. They in turn generate the s-coordinates on which our canonical form is based. We assume that there exists a set of positive integers $\kappa_1,\ldots,\kappa_m$ such that

(i) the set $C = \{g_1,(ad^1 f,g_1),\ldots,(ad^{\kappa_1-1} f,g_1),g_2,\ldots,(ad^{\kappa_2-1} f,g_2),\ldots,g_m,\ldots,(ad^{\kappa_m-1} f,g_m)\}$ spans $R^n$ on a neighborhood U near the origin,

(ii) the span of $C_j \cap C$ = the span of $C_j$, where $C_j = \{g_1,(ad^1 f,g_1),\ldots,(ad^{\kappa_j-2} f,g_1),g_2,\ldots,(ad^{\kappa_j-2} f,g_2),\ldots,g_m,\ldots,(ad^{\kappa_j-2} f,g_m)\}$ and $j=1,2,\ldots,m$,

(iii) $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_m$.

Notice that no assumption is made on the involutiveness of the system.

A crate of m columns is then formed with the vector fields in C as shown in Figure 1.

| | | | |
|---|---|---|---|
| $g_1$ | $g_2$ | $\cdots$ | $g_m$ |
| $(ad^1 f,g_1)$ | $(ad^1 f,g_2)$ | | $(ad^1 f,g_m)$ |
| $\vdots$ | $\vdots$ | | $\vdots$ |
| $(ad^{\kappa_1-1} f,g_1)$ | | | |

Figure 1.

The first column is filled from top down with $g_1,(ad^1 f,g_1),\ldots,(ad^{\kappa_1-1} f,g_1)$, the second column with $g_2,(ad^1 f,g_2),\ldots,(ad^{\kappa_2-1} f,g_2)$, and so on. Now we arrange the vector fields into a sequence in the order from left to right and from the bottom row to the top. Hence, we have

$$X_1 = (ad^{\kappa_1 - 1} f, g_1)$$

$$X_2 = \begin{cases} (ad^{\kappa_1 - 2} f, g_1), & \text{if } \kappa_1 > \kappa_2 \\ (ad^{\kappa_2 - 1} f, g_1), & \text{if } \kappa_1 = \kappa_2 \end{cases} \tag{15}$$

$$X_3 = \begin{cases} (ad^{\kappa_3 - 1} f, g_3), & \text{if } \kappa_1 = \kappa_2 = \kappa_3 \\ (ad^{\kappa_1 - 2} f, g_1), & \text{if } \kappa_1 = \kappa_2 > \kappa_3 \\ (ad^{\kappa_2 - 1} f, g_2), & \text{if } \kappa_1 = \kappa_2 + 1 \\ (ad^{\kappa_1 - 3} f, g_1), & \text{if } \kappa_1 > \kappa_2 + 1 \end{cases}$$

etc. This procedure can best be illustrated by the following example. Suppose we have a system on $R^6$ which forms a crate as shown in Figure 2. Then the sequence $X_1, x_2, \ldots, X_6$ is defined as in Figure 3.

| $g_1$ | $g_2$ | $g_3$ |
|---|---|---|
| $(ad^1 f, g_1)$ | $(ad^1 f, g_2)$ | 0 |
| $(ad^2 f, g_1)$ | 0 | 0 |

Figure 2.

| $X_4$ | $X_5$ | $X_6$ |
|---|---|---|
| $X_2$ | $X_3$ | 0 |
| $X_1$ | 0 | 0 |

Figure 3.

Using this set of linearly independent vector fields, we can then generate the s-coordinates. A straightforward application of Lemma 2 and Lemma 4 shows the following theorem.

Theorem 1. In the s-coordinates generated by the sequence of vector fields previously defined, the system equation (15) assumes the form

$$\dot{s} = f(0) + \sum_{i=1}^{\infty} \sum_{j=1}^{n} \frac{(s^j)^i}{i!} (ad^i X_j, f)\Big|_{S_{j-1}} \tag{17}$$

$$+ \sum_{k=n-m+1}^{n} \left\{ e_k + \sum_{i=1}^{\infty} \sum_{j=k+1}^{n} \frac{(s^j)^i}{i!} (ad^i X_j, X_k)\Big|_{S_{j-1}} \right\} u_{k-n+m}$$

where the $X_k$ are defined in (16).

We remark that there is an equivalent procedure for introducing the $s=(s^1, s^2, \ldots, s^n)$ coordinates. We simply solve in order the following systems of ordinary differential equations with indicated initial conditions.

$$\frac{dx(s^1)}{ds^1} = X_1(x^1, \ldots, x^n) \qquad , \; x(0) = 0 \tag{18}$$

$$\frac{dx(s^1, s^2)}{ds^2} = X_2(x^1, \ldots, x^n) \qquad , \; x(s^1, 0) = x(s^1)$$

$$\frac{dx(s^1, s^2, s^3)}{ds^3} = X_3(x^1, \ldots, x^n) \qquad , \; x(s^1, s^2, 0) = x(s^1, s^2)$$

$$\vdots$$

$$\frac{dx(s^1, s^2, \ldots, s^n)}{ds^n} = X_n(x^1, \ldots, x^n), \; x(s^1 s^2, \ldots, 0) = x(s^1, s^2, \ldots, s^{n-1}).$$

If the vector fields $X_1, X_2, \ldots, X_n$ are linearly independent, we can solve for $s^1, s^2, \ldots, s^n$ by the inverse function theorem.

For our control system (15), the vector fields $X_1, X_2, \ldots, X_n$ are defined in (16). Since the set C is assumed to span $R^n$, the s coordinates exist for our control system.

## IV. Pure Feedback Systems

As indicated in a previous discussion, the s coordinates exist under the spanning assumptions (i), (ii) and (iii) of Section 3. These assumptions are generic, in the sense that "almost all" control systems (15) satisfy them. If we also add the involutive conditions mentioned in the introduction, we obtain a nonlinear system which is feedback equivalent to a controllable linear system. With this extra assumption on involutivity, the s coordinates force the nonlinear system (15) into a special form, called a pure feedback system. Meyer and Cicolani [5] called this block triangular in a special case and showed that it is extremely easy to move from a block triangular system to a linear system (the name technique can be used to move from a pure feedback system to a linear system).

In the following definition, we consider system (15) under the spanning assumptions (i), (ii) and (iii). We form a crate for the system as described in Figure 1. Let $n_i$ be the number of elements in the $(\kappa_1-i+1)$-th row of the crate, and $\beta_k=n_1+n_2+\ldots+n_k$, where $i,k=1,2,\ldots,\kappa_1$.

<u>Definition 1</u>. The nonliner system (15) is a <u>pure feedback system</u> if it is of the form

$$\dot{x}^1 = f^1(s^1,\ldots,x^{\beta_2})$$

$$\dot{x}^2 = f^2(x^1,\ldots,x^{\beta_2})$$

$$\vdots$$

$$\dot{x}^{\beta_1} = f^{\beta_1}(x^1,\ldots,x^{\beta_2})$$

$$\dot{x}^{\beta_1+1} = f^{\beta_1+1}(x^1,\ldots,x^{\beta_3})$$

$$\dot{x}^{\beta_1+2} = f^{\beta_1+2}(x^1,\ldots,x^{\beta_3})$$

$$\vdots$$

$$\dot{x}^{\beta_2} = f^{\beta_2}(x^1,\ldots,x^{\beta_3})$$

$$\vdots$$

$$\dot{x}^{\beta_{\kappa_1-2}+1} = f^{\beta_{\kappa_1-2}+1}(x^1,\ldots,x^n)$$

$$\vdots$$

$$\dot{x}^{n-m} = f^{n-m}(x^1,\ldots,x^n)$$

$$\dot{x}^{n-m+1} = f^{n-m}(x^1,\ldots,x^n) + \sum_{i=1}^{m} g_i^{n-m+1}(x^1,\ldots,x^n)u_i$$

$$\vdots$$

$$\dot{x}^{n} = f^{n}(x^1,\ldots,x^n) + \sum_{i=1}^{m} g_i^{n}(x^1,\ldots,x^n)u_i$$

We remark that $\beta_{\kappa_1-1}=n-m$, $\beta_{\kappa_1}=n_1+\ldots+n_{\kappa_1}=n$, and $n_1\le n_2\le n_3\le\ldots\le n_{\kappa_1}$. This definition is motivated by the following diagram.

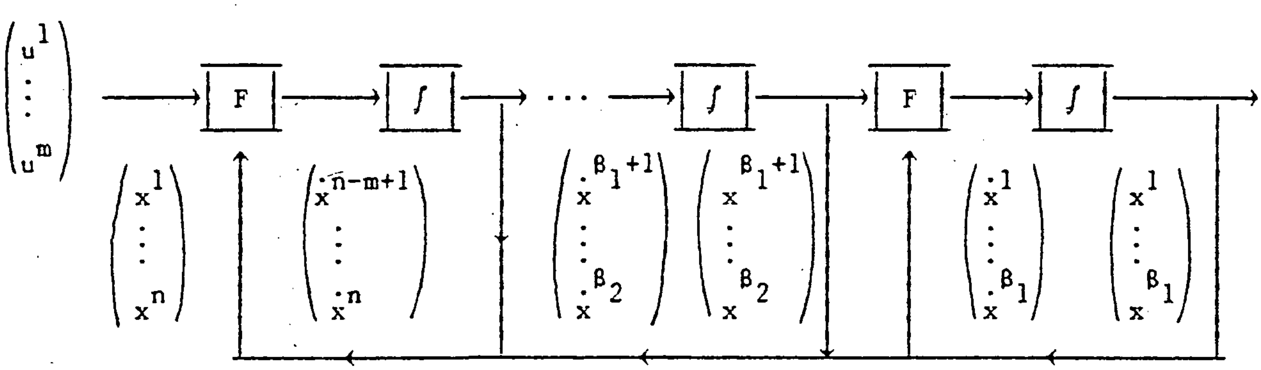


Figure 4.

We note that there is no feedforward signal flow other than the main path. Consider again a system with an associated crate as shown in Figure 2. It is a pure feedback system if it is of the form

$$\dot{x}^1 = f^1(x_1, x_2, x_3)$$

$$\dot{x}^2 = f^2(x_1, s_2, \ldots, x^6)$$

$$\dot{x}^3 = f^3(x_1, x_2, \ldots, x^6)$$

$$\dot{x}^4 = f^4(x^1, \ldots, x^6) + \sum_{i=1}^{3} g_i^4(x^1, \ldots, x^6) u_i$$

$$\dot{x}^5 = f^5(x^1, \ldots, x^6) + \sum_{i=1}^{3} g_i^5(x^1, \ldots, x^6) u_i$$

$$\dot{x}^6 = f^6(x^1, \ldots, x^6) + \sum_{i=1}^{3} g_i^6(x^1, \ldots, x^6) u_i$$

To the spanning assumptions (i), (ii) and (iii), we add

(iv) The sets $C_j = \{g_1, [f, g_1], \ldots, (ad^{\kappa_j - 2} f, g_1), g_2, [f, g_2], \ldots, (ad^{\kappa_j - 2} f, g_2), \ldots, g_m, [f, g_m], \ldots, (ad^{\kappa_j - 2} f, g_m)\}$ are involutive for $u = 1, 2, \ldots, m$.

Under this additional assumption (iv), system (15) is feedback linearizable [1]. We show that it takes a special form in the s-coordinates.

<u>Theorem 2</u>. If the sets of vector fields $C_j, j = 1, 2, \ldots, m$ are involutive, then the nonlinear system (15) is a <u>pure feedback system</u> in the s-coordinates.

<u>Proof</u>. For our nonlinear system, the vector fields $X_1, X_2, \ldots, X_n$ are defined from those elements in the set C by Equation (16). We take as before

$n_1$ = number of elements in C with ad superscript $\kappa_1-1$,

$n_2$ = number of elements in C with ad superscript $\kappa_1-2$,

$\vdots$

$n_{\kappa_1}$ = m = number of elements in C with ad superscript 0.

Also, we have $\beta_k = n_1+n_2+\ldots+n_k, 1\le k\le\kappa_1$ with $\beta_{\kappa_1} = n$. The canonical form for the system in the s-coordinates generated by $X_k$ is

$$\dot{s} = f(0) + \sum_{j=1}^{n} \sum_{i=1}^{\infty} \frac{(s^j)^i}{i!} (ad^i X_j, f) \Big|_{S_{j-1}} + \sum_{k=n-m+1}^{n} \left\{ e_k + \sum_{j=k+1}^{n} \sum_{i=1}^{\infty} \frac{(s^j)^i}{i!} (ad^i X_j, X_k)\Big|_{S_{j-1}} \right\} u_{k-m+m} \tag{17}$$

It has been shown in [1] that the involutivity of $C_j, j=1,2,\ldots,m$ implies that each of the sets of vector fields $\{X_{\beta_k}+1,\ldots,X_m\}, k=1,2,\ldots,\kappa_1-1$ is involutive.

In particular, when $k=\kappa_1-1$, the set $\{X_{\beta_k}+1,\ldots,X_m\} = \{g_1,\ldots,g_m\}$ is involutive. From Lemma 3, this implies that the control terms in (17) is of the form

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{pmatrix} U_1 + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{pmatrix} U_2 + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{pmatrix} U_m \tag{20}$$

where * means possibly nonvanishing functions of $s_1, s_2, \ldots, s_m$, and they begin at $(\beta_{\kappa_1-1}+1)$-th entry.

The second summation is rewritten as

$$\sum_{j=1}^{\beta_1} \sum_{i=1}^{\infty} \frac{(s^j)^i}{i!} (ad^i X_j, f)\Big|_{S_{j-1}} \tag{21}$$

$$+ \sum_{j=\beta_1+1}^{\beta_2} \sum_{i=1}^{\infty} \frac{(s^j)^i}{i!} (ad^i X_j, f)\Big|_{S_{j-1}}$$

$$\vdots$$

$$+ \sum_{j=\beta_{\kappa_1-1}+1}^{\beta_{\kappa_1}} \sum_{i=1}^{\infty} \frac{(s^j)^i}{i!} (ad^i X_j, f)\Big|_{S_{j-1}}$$

Observe that as the index j runs from $\beta_{\kappa_1-1}+1$ to $\beta_{\kappa_1}=n$ in the last term of the summation (21), the vector fields $(ad^1 X_j, f)$ are either simply $X_k, \beta_{\kappa_1-2}+1 \le k \le \beta_{\kappa_1-1}$ or linear dependents on $X_\ell, \beta_{\kappa_1-2}+1 \le \ell \le \beta_{\kappa_1}$. Further, because the set $\{X_{\beta_{\kappa_1-2}+1}, \ldots, X_{\beta_{\kappa_1}}=X_n\}$ is involutive, it is easy to see that

$$(ad^i X_j, f) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{pmatrix} \leftarrow (\beta_{\kappa_1-2}+1)\text{-th place.} \tag{22}$$

When restricted to $S_{j-1}$, the asterisks in (22) are functions of $s^1, \ldots, s^{n-1}$. As we sum up the series, we have

$$\sum_{j=\beta_{\kappa_1-1}+1}^{\beta_{\kappa_1}} \sum_{i=1}^{\infty} \frac{(s^j)^i}{i!} (ad^i X_j, f)\Big|_{S_{j-1}}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{pmatrix} \leftarrow (\beta_{\kappa_1-2}+1)\text{-th place}$$

where $*$ are functions of $s^1, s^2, \ldots, s^n$.

Continuing the analysis with every term in summation (21), it is seen that

$$(ad^i X_j, f) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{pmatrix} \leftarrow (\beta_{\kappa_1 - k}+1)\text{-th place} \tag{23}$$

when the term with j running from $\beta_{\kappa_1-k}+1$ to $\beta_{\kappa_1-k+1}$ is considered where $k=1,2,\ldots,\kappa_1$, and $\beta_0=0$. In this case, $S_{j-1}$ is the manifold characterized by $s^j=s^{j+1}=\ldots=s^n=0$. Thus, the asterisks in (23), when restricted to $S_{j-1}$, are functions of $s^1,\ldots,s^{j-1}$. Summing up the subseries, we have

$$\sum_{j=\beta_{\kappa_1-k}+1}^{\beta_{\kappa_1-k}+1} \sum_{i=1}^{\infty} \frac{(s^j)^i}{i!} (ad^i X_j, f)\Big|_{S_j}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{pmatrix} \leftarrow (\beta_{\kappa_1 - k}+1)\text{-th place}$$

where $*$ are functions of $s^1, s^2, \ldots, s^{\beta_{\kappa_1-k}+1}$, but is independent of $s^\ell$, $\ell=\beta_{\kappa_1-k+1}+1,\ldots,n$.

This concludes our claim that the system is a pure feedback system. □

We remark that for this paper we have worked in a neighborhood of the origin in $R^n$, but any point $x_0$ in $R^n$ where the system satisfies the spanning conditions could have been considered.

## V. Conclusions and Related Research

For a real analytic system of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \qquad , \; x \varepsilon R^n$$

we have introduced a coordinate system which reflects the intrinsic differential geometry of the system. The basic vectors of the s-coordinate systems are generated by the vector fields in the set C, and the indices $\kappa_1, \kappa_2, \ldots, \kappa_m$ correspond to Kronecker indices for a controllable linear system. Moreover, we have introduced a "canonical expansion" of the nonlinear system in the s-coordinates, and the usual Taylor series terms are replaced by a Lie bracket expansion. Under involutive assumptions on subsets of the vector fields in C, we show the nonlinear system is a pure feedback system in the s-coordinates.

In general, a "measure of how close" a nonlinear system is to a pure feedback system should be indicated in its canonical expansion in the s-coordinates. This is especially interesting since pure feedback systems are feedback equivalent to controllable linear systems.

It is known that in the s-coordinates the usual Taylor series linear approximation to a nonlinear system and a linear approximation generated by certain Lie brackets agree [6]. Adding outputs to a nonlinear system allows us to discuss Volterra series expansions. Since Volterra series kernels involve Lie derivatives and Lie brackets [7], the s-coordinates may be natural for such series. Moreover, the s-coordinates may prove advantageous for nonlinear input-output systems realization.

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